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Global dynamics of SIS models with transport-related infection

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Abstract

To understand the effect of transport-related infection on disease spread, an epidemic model for several regions which are connected by transportation of individuals has been proposed by Cui, Takeuchi and Saito [J. Cui, Y. Takeuchi, Y. Saito, Spreading disease with transport-related infection, J. Theoret. Biol. 239 (2006) 376–390]. Transportation among regions is one of the main factors which affects the outbreak of diseases. The purpose of this paper is the further study of the local asymptotic stability of the endemic equilibrium and the global dynamics of the system. Sufficient conditions are established for global asymptotic stability of the endemic equilibrium. Permanence is also discussed. It is shown that the disease is endemic in the sense of permanence if and only if the endemic equilibrium exists. This implies that transport-related infection on disease can make the disease endemic even if all the isolated regions are disease free.

Keywords: Transport-related infection; Basic reproduction number; Stability; Permanence

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1. Introduction

Population dispersal may cause virus transmission from one region to other regions. For example, the arrival of new infectives has been demonstrated as an important factor in the outbreaks of measles observed in Iceland (see Cliff, Haggett and Smallman-Raynor [3]). Recently, some epidemic models have been proposed to understand the spread dynamics of infectious disease.

Discrete time difference equations in a continuous state space were used by Rvachev and Longini (Longini [8], Rvachev and Longini [9]) to study the global spread of influenza, taking into account the airline network. Sattenspiel and Dietz [10] introduced a model with travel between populations. They identified parameters in the case of the transmission of measles in the Caribbean island of Dominica, and numerically studied the dynamics of the model. Sattenspiel and Herring [11] considered the same type of the model but applied it to travel between populations in the Canadian subarctic, which can be thought of as a closed population where travel is easily quantified. Recently, the same authors (Sattenspiel and Herring [12]) formulated a model that includes quarantine and applied it to data of the 1918–1919 influenza epidemic in the center of Canada. Fulford et al. (Fulford, Roberts and Heesterbeek [5]) have formulated a metapopulation model with age-structure. Wang and Mulone [15] and Wang and Zhao [16,17] have proposed epidemic models to describe the dynamics of disease spread between two patches and n patches. Arino and van den Driessche [1] have also developed a multi-city epidemic model to analyze the spatial spread of infectious diseases.

All the above studies ignore the possibility for the individuals to become infective during travel. Recently, Cui, Takeuchi and Saito [4] have proposed an epidemic model to understand the effect of transport-related infection on disease spread. In their paper, Cui, Takeuchi and Saito [4] proposed the following SIS model:

$$\begin{aligned}
\dot{S}_1 &= a - \frac{\beta S_1 I_1}{S_1 + I_1} - bS_1 + dI_1 - \alpha S_1 + \alpha S_2 - \frac{\gamma\alpha S_2 I_2}{S_2 + I_2},\\
\dot{I}_1 &= \frac{\beta S_1 I_1}{S_1 + I_1} - (c + d + \alpha)I_1 + \alpha I_2 + \frac{\gamma\alpha S_2 I_2}{S_2 + I_2},\\
\dot{S}_2 &= a - \frac{\beta S_2 I_2}{S_2 + I_2} - bS_2 + dI_2 - \alpha S_2 + \alpha S_1 - \frac{\gamma\alpha S_1 I_1}{S_1 + I_1},\\
\dot{I}_2 &= \frac{\beta S_2 I_2}{S_2 + I_2} - (c + d + \alpha)I_2 + \alpha I_1 + \frac{\gamma\alpha S_1 I_1}{S_1 + I_1}.
\end{aligned} \tag{1.1}$$

Here S_i and I_i represent the number of susceptible and infected individuals in city i, respectively ($i = 1, 2$). In this model, we adopt the fixed number of offspring, denoted by a, joins into the susceptible class per unit time. Natural death rate for susceptible individuals is a constant per capita rate b. Infected individuals recover at a constant per capita rate d, and the per capita mortality rate for infected individual is c. Since this includes both natural and disease induced mortality, we have $c \geqslant b$.

Disease is transmitted with the incidence rate (that is, the number of new cases of infection per unit time)

$$\frac{\beta S_j I_j}{S_j + I_j}, \quad j = 1, 2,$$

within city j. The transmission rate within a city is a constant β. Susceptible and infected individuals of every city i leave to city j ($j \neq i$, $i, j = 1, 2$) at a per capita rate α. When the individuals in city j travel to city i, disease is transmitted with the incidence rate

$$\frac{\gamma(\alpha S_j)(\alpha I_j)}{\alpha S_j + \alpha I_j} = \frac{\gamma \alpha S_j I_j}{S_j + I_j} \quad (j = 1, 2)$$

with a transmission rate $\gamma\alpha$. We assume that both cities are identical, i.e. demographic parameters are the same for each city.

From the biological point of view, the number of the susceptibles during travel should be nonnegative, that is, for $j = 1, 2$ and for all $S_j, I_j \geqslant 0$,

$$\alpha S_j - \frac{\gamma \alpha S_j I_j}{S_j + I_j} \geqslant 0.$$

This property is satisfied if $0 \leqslant \gamma \leqslant 1$, which we assume in this paper. Also note that the condition $0 \leqslant \gamma \leqslant 1$ ensures that any solution of (1.1) is nonnegative if its initial value is nonnegative.

Considering entry screening and exit screening to detect infected individuals (with probability of successfully detecting an infected individual θ_e, θ_d, $0 \leqslant \theta_e, \theta_d \leqslant 1$, respectively), Liu and Takeuchi [7] proposed an SIQS model and mathematically studied the following special case of $\theta_e = \theta, \theta_d = 0$:

$$\begin{aligned}
\dot{S}_1 &= a - \frac{\beta S_1 I_1}{S_1 + I_1} - b S_1 + d I_1 + f Q_1 - \alpha S_1 + \alpha S_2 - \frac{\gamma \alpha S_2 I_2}{S_2 + I_2},\\
\dot{I}_1 &= \frac{\beta S_1 I_1}{S_1 + I_1} - (c + d + \alpha) I_1 + (1 - \theta)\alpha I_2 + (1 - \theta)\frac{\gamma \alpha S_2 I_2}{S_2 + I_2},\\
\dot{Q}_1 &= \theta \alpha I_2 + \theta \frac{\gamma \alpha S_2 I_2}{S_2 + I_2} - (e + f) Q_1,\\
\dot{S}_2 &= a - \frac{\beta S_2 I_2}{S_2 + I_2} - b S_2 + d I_2 + f Q_2 - \alpha S_2 + \alpha S_1 - \frac{\gamma \alpha S_1 I_1}{S_1 + I_1},\\
\dot{I}_2 &= \frac{\beta S_2 I_2}{S_2 + I_2} - (c + d + \alpha) I_2 + (1 - \theta)\alpha I_1 + (1 - \theta)\frac{\gamma \alpha S_1 I_1}{S_1 + I_1},\\
\dot{Q}_2 &= \theta \alpha I_1 + \theta \frac{\gamma \alpha S_1 I_1}{S_1 + I_1} - (e + f) Q_2.
\end{aligned} \tag{1.2}$$

If $\theta = 0$, obviously, $Q_i \to 0$, $i = 1, 2$, as $t \to \infty$. Hence model (1.1) is exactly the limit system of model (1.2) on the invariant subset $\Omega_0 = \{(S_1, I_1, 0, S_2, I_2, 0) \mid S_i \geqslant 0,\ I_i \geqslant 0,\ i = 1, 2\}$ of $\Re^6_+$.

Mathematically, Cui, Takeuchi and Saito [4] mainly studied local asymptotic stability of model (1.1) and the endemic equilibrium was proved to be asymptotically stable with an additional condition besides the condition for its existence. Liu and Takeuchi [7] studied local asymptotic stability and permanence of model (1.2) and proved that the endemic equilibrium is asymptotically stable if it exists and that the disease is endemic in the sense of permanence. In Liu and Takeuchi [7], the global asymptotic stability of equilibria remains unsolved. The global dynamics of a system is far from known with local asymptotic stability since it gives only dynamics of the solutions near the equilibrium. Permanence may give a rough global dynamic of all the positive solutions. However, with global asymptotic stability, the dynamics of a system is completely clear.

The purpose of this paper is to study further the global dynamics of model (1.1). The local asymptotic stability of the endemic equilibrium point is discussed in Section 2, and its global asymptotic stability and the permanence of system (1.1) are studied in Section 3. In the final section, we will discuss our results.

2. Local dynamics

Cui, Takeuchi and Saito [4] defines two numbers as

$$\Re_0 = \frac{\beta}{c+d}, \qquad \Re_{0\gamma} = \Re_0 + \frac{\gamma\alpha}{c+d}, \tag{2.1}$$

where $\Re_{0\gamma}$ is the basic reproduction number for (1.1), $\Re_0$ is the basic reproduction number of the following model:

$$\begin{aligned} \dot{S} &= a - \frac{\beta SI}{S+I} - bS + dI, \\ \dot{I} &= \frac{\beta SI}{S+I} - (c+d)I. \end{aligned} \tag{2.2}$$

It follows from Cui, Takeuchi and Saito [4] that system (1.1) has a disease free equilibrium $E_0(\frac{a}{b}, 0, \frac{a}{b}, 0)$ for all parameter values, and an endemic equilibrium $E_+(S_\gamma^*, I_\gamma^*, S_\gamma^*, I_\gamma^*)$ appears in two cities when $\Re_{0\gamma} > 1$. They have shown the following result.

Lemma 2.1. *For model* (1.1)*, the disease free equilibrium E_0 is locally asymptotically stable provided $\Re_{0\gamma} < 1$. When $\Re_{0\gamma} > 1$, E_0 is unstable and the endemic equilibrium $E_+(S_\gamma^*, I_\gamma^*, S_\gamma^*, I_\gamma^*)$ appears in both cities, where*

$$S_\gamma^* = \frac{a}{b + c(\Re_{0\gamma} - 1)}, \qquad I_\gamma^* = \frac{a(\Re_{0\gamma} - 1)}{b + c(\Re_{0\gamma} - 1)}. \tag{2.3}$$

The endemic equilibrium E_+ is locally asymptotically stable if $1 < \Re_{0\gamma} \leqslant 2\Re_0$.

Lemma 2.1 shows the possibility that the entire region becomes endemic even if each isolated city is disease free. In fact, let us assume that $\Re_0 < 1$ and $1 < \Re_{0\gamma} \leqslant 2\Re_0$. The former assumption implies that the isolated city is disease free, that is, the equilibrium point $E_0 = (a/b, 0)$ of (2.2) is globally asymptotically stable. If $1 > \Re_0 > 1/2$, then it is possible to choose the transmission rate of disease during travel $\gamma\alpha$ satisfying the second assumption $1 < \Re_{0\gamma} \leqslant 2\Re_0$. Lemma 2.1 shows that the endemic equilibrium E_+ of (1.1) is locally asymptotically stable, even if the disease free equilibrium of (2.2) is globally asymptotically stable.

In this section we further study the local asymptotic stability of E_+ only assuming $\Re_{0\gamma} > 1$ and generalize Lemma 2.1.

Theorem 2.2. *For model* (1.1)*, the endemic equilibrium $E_+(S_\gamma^*, I_\gamma^*, S_\gamma^*, I_\gamma^*)$ is locally asymptotically stable when $\Re_{0\gamma} > 1$.*

Proof. By (1.1) and (2.3), or by Appendix C of Cui, Takeuchi and Saito [4], the Jacobian matrix evaluated at E_+ is

$$J(E_+) = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} -\beta(1-\frac{1}{\Re_{0\gamma}})^2 - b - \alpha & -\frac{\beta}{\Re_{0\gamma}^2} + d \\ \beta(1-\frac{1}{\Re_{0\gamma}})^2 & \frac{\beta}{\Re_{0\gamma}^2} - (c+d+\alpha) \end{pmatrix},$$

$$B = \begin{pmatrix} \alpha - \gamma\alpha(1-\frac{1}{\Re_{0\gamma}})^2 & -\frac{\gamma\alpha}{\Re_{0\gamma}^2} \\ \gamma\alpha(1-\frac{1}{\Re_{0\gamma}})^2 & \alpha + \frac{\gamma\alpha}{\Re_{0\gamma}^2} \end{pmatrix}.$$

By Appendix C in Cui, Takeuchi and Saito [4], the characteristic polynomial of $J(E_+)$ can be calculated as

$$\det\bigl(J(E_+) - \lambda I\bigr) = \det(A + B - \lambda I)\det(A - B - \lambda I)$$

and if $\Re_{0\gamma} > 1$, $0 \leqslant \gamma \leqslant 1$, the following properties always hold:

(i) $\operatorname{tr}(A+B) < 0$, $\det(A+B) > 0$.
(ii) $\operatorname{tr}(A-B) < 0$.

Hence it suffices to prove that $\det(A-B) > 0$ also holds when $\Re_{0\gamma} > 1$, $0 \leqslant \gamma \leqslant 1$. By (2.1), we have

$$(c+d) = (\beta + \gamma\alpha)\frac{1}{\Re_{0\gamma}} = (\beta - \gamma\alpha + 2\gamma\alpha)\frac{1}{\Re_{0\gamma}} \tag{2.4}$$

and

$$(\beta - \gamma\alpha) = (2\Re_0 - \Re_{0\gamma})(c+d). \tag{2.5}$$

Since

$$A - B = \begin{pmatrix} -(\beta-\gamma\alpha)(1-\frac{1}{\Re_{0\gamma}})^2 - b - 2\alpha & -(\beta-\gamma\alpha)\frac{1}{\Re_{0\gamma}^2} + d \\ (\beta-\gamma\alpha)(1-\frac{1}{\Re_{0\gamma}})^2 & (\beta-\gamma\alpha)\frac{1}{\Re_{0\gamma}^2} - (c+d+2\alpha) \end{pmatrix},$$

by adding the second row to the first row, we get

$$C \triangleq \begin{pmatrix} -(b+2\alpha) & -(c+2\alpha) \\ (\beta-\gamma\alpha)(1-\frac{1}{\Re_{0\gamma}})^2 & (\beta-\gamma\alpha)\frac{1}{\Re_{0\gamma}^2} - (c+d+2\alpha) \end{pmatrix}.$$

Therefore

$$\begin{aligned} \det(A-B) &= \det(C) \\ &= (b+2\alpha)(c+d+2\alpha) - (b+2\alpha)(\beta-\gamma\alpha)\frac{1}{\Re_{0\gamma}^2} \\ &\quad + (c+2\alpha)(\beta-\gamma\alpha)\left(1 - \frac{1}{\Re_{0\gamma}}\right)^2. \end{aligned} \tag{2.6}$$

We discuss the sign of $\det(A-B)$ as the following two cases.

Case 1. $\Re_{0\gamma} \leqslant 2\Re_0$.

By (2.4), (2.6) can be rewritten as

$$\begin{aligned}\det(A-B) &= \det(C)\\ &= (b+2\alpha)\left(2\gamma\alpha\frac{1}{\Re_{0\gamma}}+2\alpha\right)+(\beta-\gamma\alpha)(b+2\alpha)\frac{1}{\Re_{0\gamma}}\left(1-\frac{1}{\Re_{0\gamma}}\right)\\ &\quad +(\beta-\gamma\alpha)(c+2\alpha)\left(1-\frac{1}{\Re_{0\gamma}}\right)^2.\end{aligned}$$

By (2.5), we have $\det(A-B)>0$.

Case 2. $\Re_{0\gamma}>2\Re_0$.
(2.6) can be rewritten as

$$\begin{aligned}\det(A-B) &= \det(C)\\ &= (b+2\alpha)d+(c+2\alpha)\left\{b+\beta\left(1-\frac{1}{\Re_{0\gamma}}\right)^2+\left(2-\gamma\left(1-\frac{1}{\Re_{0\gamma}}\right)^2\right)\alpha\right\}\\ &\quad -(b+2\alpha)(\beta-\gamma\alpha)\frac{1}{\Re_{0\gamma}^2}.\end{aligned}$$

By (2.5) and $\Re_{0\gamma}>1$, $0\leqslant\gamma\leqslant 1$, we can also observe that $\det(A-B)>0$. Hence E_+ is locally asymptotically stable if $\Re_{0\gamma}>1$. This completes the proof. □

Remark 2.1. The result of Theorem 2.2 may be obtained by using the result of Theorem 2.2 in Liu and Takeuchi [7] and considering model (1.1) as a limit system of model (1.2). To avoid using limit system, we give here a direct and simple proof, which improves the result and proof of the Theorem in Cui, Takeuchi and Saito [4].

3. Global dynamics

Firstly, we consider permanence of the disease. Set initial conditions as $S_i(0)\geqslant 0$ and $I_i(0)\geqslant 0$ for $i=1,2$. It is easy to check that all solutions of (1.1) are nonnegative (that is, $S_i(t)\geqslant 0$ and $I_i(t)\geqslant 0$ for $t\geqslant 0$ and $i=1,2$) under the assumption $0\leqslant\gamma\leqslant 1$. The following result shows that system (1.1) is ultimately bounded above. Since its proof is similar to the proof of Theorem 3.1 in Liu and Takeuchi [7] with $V(t)=S_1(t)+I_1(t)+S_2(t)+I_2(t)$ and $m=\min\{b,c\}>0$, it will be omitted.

Lemma 3.1. *Let* $(S_1(t),I_1(t),S_2(t),I_2(t))$ *be the solution of* (1.1) *with initial values* $S_i(0)\geqslant 0$ *and* $I_i(0)\geqslant 0$ *for* $i=1,2$. *Then there exist* $M>0$ *and* $t_1>0$ *such that* $S_i(t)\leqslant M$ *and* $I_i(t)\leqslant M$ *for* $i=1,2$ *and* $t\geqslant t_1$.

Now we can study the permanence of system (1.1).

Theorem 3.2. *Let* $\Re_{0\gamma}>1$. *Then there exists* $\varepsilon>0$ *such that every solution* $(S_1(t),I_1(t),S_2(t),I_2(t))$ *of* (1.1) *with initial values* $S_i(0)\geqslant 0$, $I_i(0)\geqslant 0$ *for* $i=1,2$ *and* $I_1(0)+I_2(0)>0$ *satisfies*

$$\liminf_{t\to\infty}S_i(t)\geqslant\varepsilon,\qquad\liminf_{t\to\infty}I_i(t)\geqslant\varepsilon,\qquad i=1,2.$$

Proof. The result follows from an application of Theorem 4.6 in Thieme [14]. Define

$$X = \{(S_1, I_1, S_2, I_2) \mid S_i \geqslant 0,\ I_i \geqslant 0,\ i = 1, 2\},$$
$$X_0 = \{(S_1, I_1, S_2, I_2) \in X \mid I_1 + I_2 > 0\},$$
$$\partial X_0 = X \setminus X_0.$$

It then suffices to show that (1.1) is uniformly persistent with respect to $(X_0, \partial X_0)$.

It is easy to verify that both X and X_0 are positively invariant with respect to system (1.1). Denote the omega limit set of the solution of system (1.1) starting in $(S_1(0), I_1(0), S_2(0), I_2(0)) \in X$ by $\omega(S_1(0), I_1(0), S_2(0), I_2(0))$ (which exists by Lemma 3.1). Let

$$M_\partial = \{(S_1(0), I_1(0), S_2(0), I_2(0)) \mid (S_1(t), I_1(t), S_2(t), I_2(t)) \in \partial X_0,\ \forall t \geqslant 0\}$$

and

$$\Omega = \bigcup \{\omega(S_1(0), I_1(0), S_2(0), I_2(0)) \mid (S_1(0), I_1(0), S_2(0), I_2(0)) \in M_\partial\}.$$

We can prove that

$$M_\partial = \{(S_1, 0, S_2, 0) \mid S_i \geqslant 0,\ i = 1, 2\} \quad \text{and} \quad \Omega = \{E_0\},$$

where $E_0 = (\bar{S}, 0, \bar{S}, 0)$ and $\bar{S} = a/b$. Then as the proof of Lemma 3.5 in Leenheer and Smith [6], we can finish the proof by showing that E_0 is an isolated and acyclic covering of Ω, and is a weak repeller for X_0. Since these are similar to the case of $0 \leqslant \theta < 1$ in the proof of Theorem 3.5 in [7], the details will be omitted here. The proof is complete. □

Remark 3.1. If $I_1(0) + I_2(0) > 0$ is replaced by $I_i(0) > 0$ for $i = 1, 2$, then the result may also be obtained from Theorem 3.5 in Liu and Takeuchi [7] by considering model (1.2) on $\Omega_0 = \{(S_1, I_1, 0, S_2, I_2, 0) \mid S_i \geqslant 0,\ I_i \geqslant 0,\ i = 1, 2\}$ with $\theta = 0$. However, from Theorem 3.2, we can see that the conclusion of [7, Theorem 3.5] was not stated clearly. It should be that every solution $(S_1(t), I_1(t), Q_1(t), S_2(t), I_2(t), Q_2(t))$ of (1.2) with initial values $S_i(0) \geqslant 0$, $I_i(0) > 0$ and $Q_i(0) \geqslant 0$ satisfies

$$\liminf_{t\to\infty} S_i(t) \geqslant \varepsilon, \quad \liminf_{t\to\infty} I_i(t) \geqslant \varepsilon, \quad \liminf_{t\to\infty} Q_i(t) \geqslant \varepsilon, \quad i = 1, 2 \text{ for } 0 < \theta \leqslant 1$$

and

$$\liminf_{t\to\infty} S_i(t) \geqslant \varepsilon, \quad \liminf_{t\to\infty} I_i(t) \geqslant \varepsilon, \quad \lim_{t\to\infty} Q_i(t) = 0, \quad i = 1, 2 \text{ for } \theta = 0.$$

Then since Ω_0 is invariant when $\theta = 0$, we will have

$$\liminf_{t\to\infty} S_i(t) \geqslant \varepsilon, \quad \liminf_{t\to\infty} I_i(t) \geqslant \varepsilon, \quad Q_i(t) \equiv 0, \quad i = 1, 2,$$

which is exactly the result of Theorem 3.2. Similarly, [7, Theorem 3.5] can also be improved for $I_1(0) + I_2(0) > 0$ instead of $I_i(0) > 0$ for $i = 1, 2$.

Lemma 3.1 and Theorem 3.2 imply that system (1.1) is permanent if $\Re_{0\gamma} > 1$.

Next, we consider the global asymptotic stability of E_+. The sets X and X_0 defined in the proof of Theorem 3.2 will still be used.

Theorem 3.3. *Suppose that*

$$|\beta - \gamma\alpha| < \frac{4(b + 2\alpha)(c + d + 2\alpha) - d^2}{4(2d + b + 4\alpha + c)}. \tag{3.1}$$

Then the disease free equilibrium point E_0 *of* (1.1) *is globally asymptotically stable on* X *for* $\Re_{0\gamma} \leqslant 1$. *The endemic equilibrium point* E_+ *is globally asymptotically stable on* X_0 *for* $\Re_{0\gamma} > 1$.

Proof. Let us consider the function:

$$V(t) = \left\{ \left(S_1(t) - S_2(t)\right)^2 + \left(I_1(t) - I_2(t)\right)^2 \right\}/2.$$

The time derivative of $V(t)$ along the solution of (1.1) becomes

$$\begin{aligned}
\dot{V} &= (\dot{S}_1 - \dot{S}_2)(S_1 - S_2) + (\dot{I}_1 - \dot{I}_2)(I_1 - I_2) \\
&= \left\{ (\beta - \gamma\alpha)\left(\frac{S_2 I_2}{S_2 + I_2} - \frac{S_1 I_1}{S_1 + I_1} \right) - (b + 2\alpha)(S_1 - S_2) + d(I_1 - I_2) \right\}(S_1 - S_2) \\
&\quad + \left\{ (\beta - \gamma\alpha)\left(\frac{S_1 I_1}{S_1 + I_1} - \frac{S_2 I_2}{S_2 + I_2} \right) - (c + d + 2\alpha)(I_1 - I_2) \right\}(I_1 - I_2) \\
&= -(b + 2\alpha)(S_1 - S_2)^2 - (c + d + 2\alpha)(I_1 - I_2)^2 + d(I_1 - I_2)(S_1 - S_2) \\
&\quad + (\beta - \gamma\alpha)\left(\frac{S_2 I_2}{S_2 + I_2} - \frac{S_1 I_1}{S_1 + I_1} \right)(S_1 - S_2 - I_1 + I_2).
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{S_2 I_2}{S_2 + I_2} - \frac{S_1 I_1}{S_1 + I_1} &= \frac{1}{(S_1 + I_1)(S_2 + I_2)} \left\{ S_1 S_2 (I_2 - I_1) + I_1 I_2 (S_2 - S_1) \right\} \\
&\leqslant |I_2 - I_1| + |S_2 - S_1|,
\end{aligned}$$

which gives the following

$$\begin{aligned}
\dot{V} &\leqslant -(b + 2\alpha)(S_1 - S_2)^2 - (c + d + 2\alpha)(I_1 - I_2)^2 + d|I_1 - I_2||S_1 - S_2| \\
&\quad + |\beta - \gamma\alpha| \left(|S_1 - S_2| + |I_1 - I_2| \right)^2 \\
&= -\left\{ (b + 2\alpha) - |\beta - \gamma\alpha| \right\} |S_1 - S_2|^2 - \left\{ (c + d + 2\alpha) - |\beta - \gamma\alpha| \right\} |I_1 - I_2|^2 \\
&\quad + \left(d + 2|\beta - \gamma\alpha| \right) |S_1 - S_2||I_1 - I_2|.
\end{aligned}$$

The above quadratic form is negative definite if and only if

$$\begin{aligned}
&|\beta - \gamma\alpha| < b + 2\alpha, \qquad |\beta - \gamma\alpha| < c + d + 2\alpha, \\
&\left(d + 2|\beta - \gamma\alpha| \right)^2 < 4\left(b + 2\alpha - |\beta - \gamma\alpha| \right)\left((c + d + 2\alpha) - |\beta - \gamma\alpha| \right).
\end{aligned}$$

It is easy to check that the above conditions are satisfied if and only if (3.1) is satisfied. Hence we can find some positive constant λ satisfying

$$\dot{V} \leqslant -\lambda \left\{ (S_1 - S_2)^2 + (I_1 - I_2)^2 \right\} = -2\lambda V(t),$$

which shows that for any solution $(S_1(t), I_1(t), S_2(t), I_2(t))$ of (1.1), we have

$$\lim_{t \to \infty} \left\{ S_1(t) - S_2(t) \right\} = 0, \qquad \lim_{t \to \infty} \left\{ I_1(t) - I_2(t) \right\} = 0.$$

By Lyapunov's theorem, we know that $\omega(x) \cap \Re_+^4$ is contained in the set $\{x \in \Re_+^4 \mid \dot{V} = 0\} = \{(S_1, I_1, S_2, I_2) \in \Re_+^4 \mid S_1 = S_2,\ I_1 = I_2\}$. Here $\omega(x)$ is the ω-limit set of the solution of (1.1) with an initial value x and $\Re_+^4$ is the state space $\{x \in \Re^4 \mid x \geqslant 0\}$. From the discussion of Cui, Takeuchi and Saito [4], we know that on $\Omega_1 = \{(S_1, I_1, S_2, I_2) \mid S_1 = S_2 > 0,\ I_1 = I_2 > 0\}$, the disease free equilibrium point E_0 of (1.1) is globally asymptotically stable for $\Re_{0\gamma} = (\beta + \gamma\alpha)/$

$(c+d) \leqslant 1$ and the endemic equilibrium point E_+ is globally asymptotically stable for $\Re_{0\gamma} = (\beta+\gamma\alpha)/(c+d) > 1$. Then the results can be obtained directly by considering (1.1) on Ω_1 as a limit system of (1.1) and using the result of Castillo-Chavez and Thieme [2, Theorem 2.3]. This completes the proof. $\square$

Remark 3.2. Theorem 3.3 extends the global asymptotic stability (GAS) results obtained in Cui, Takeuchi and Saito [4] in the following sense:

(i) Cui, Takeuchi and Saito [4] show GAS when we restrict the initial values to the set $\Omega_1 = \{(S_1, I_1, S_2, I_2) \mid S_1 = S_2 > 0,\ I_1 = I_2 > 0\}$ (GAS with respect to Ω_1). Theorem 3.3 shows GAS with respect to $X = \Re^4_+$ for E_0 and with respect to $X_0 \supseteq \mathrm{int}(\Re^4_+)$ for E_+, under the condition (3.1).
(ii) Also Cui, Takeuchi and Saito [4] show GAS under the condition $\beta = \gamma\alpha$. Note that this condition is included in (3.1).

From its proof, we know that Theorem 3.3 considers the case where the differences between susceptible populations or infected populations in two cities will disappear as time tends to infinity. Now let us consider the special case $b = c$ where the disease does not give the additional death to the infected individuals. The following lemma shows that now the differences between the total populations in two cities will disappear.

Lemma 3.4. *If $b = c$, then the total population in each city tends to a/b as $t \to \infty$.*

Proof. Define the difference of populations in both cities as $D = (S_1 + I_1) - (S_2 + I_2)$. Then we have

$$\dot{D} = -(2\alpha + b)D,$$

which shows that $D(t) \to 0$ as $t \to \infty$.

The total population $T = (S_1 + I_1) + (S_2 + I_2)$ of both cities satisfies that

$$\dot{T} = 2a - bT,$$

which shows that $T(t) \to 2a/b$ as $t \to \infty$. This completes the proof. $\square$

Now let us consider (1.1) on the subset $\bar{\Omega} = \{(S_1, I_1, S_2, I_2) \mid S_i + I_i = a/b\ (i = 1, 2)\} \subset \Re^4_+$. (1.1) can be expressed as

$$\begin{aligned}
\dot{I}_1 &= -(c+d+\alpha-\beta)I_1 + \alpha(1+\gamma)I_2 - \frac{b}{a}\beta I_1^2 - \frac{b}{a}\gamma\alpha I_2^2, \\
\dot{I}_2 &= \alpha(1+\gamma)I_1 - (c+d+\alpha-\beta)I_2 - \frac{b}{a}\gamma\alpha I_1^2 - \frac{b}{a}\beta I_2^2.
\end{aligned} \tag{3.2}$$

Note that (3.2) is symmetric with respect to the parameters and its equilibrium points must satisfy $I_1 = I_2$. This observation gives that (3.2) has a unique positive equilibrium point $E_+ = (I^*, I^*)$ if and only if $\Re_{0\gamma} > 1$, where

$$I^* = \frac{a}{b}\left(1 - \frac{c+d}{\beta+\gamma\alpha}\right) > 0.$$

Now consider the Jacobian J of (3.2) evaluated at E_+, where

$$J = \begin{pmatrix} -(c+d+\alpha-\beta) - \frac{2b}{a}\beta I^* & \alpha(1+\gamma) - \frac{2b}{a}\gamma\alpha I^* \\ \alpha(1+\gamma) - \frac{2b}{a}\gamma\alpha I^* & -(c+d+\alpha-\beta) - \frac{2b}{a}\beta I^* \end{pmatrix}.$$

The trace of J is given by

$$\frac{1}{2}\operatorname{tr} J = -(c+d+\alpha-\beta) - \frac{2b}{a}\beta I^* = -(c+d+\alpha+\beta) + 2\beta\frac{c+d}{\beta+\gamma\alpha},$$

which is negative if and only if

$$(\beta+\gamma\alpha)(c+d+\alpha+\beta) > 2\beta(c+d).$$

Define a function $f(\beta)$ of β as

$$f(\beta) = \beta^2 - \bigl(c+d-\alpha(1+\gamma)\bigr)\beta + \gamma\alpha(c+d+\alpha).$$

It is easy to check $\operatorname{tr} J < 0$ if and only if $f(\beta) > 0$. Since

$$c+d-\gamma\alpha > \bigl(c+d-\alpha(1+\gamma)\bigr)/2, \qquad f(c+d-\gamma\alpha) = \alpha(1+\gamma)(c+d) > 0,$$

$\operatorname{tr} J < 0$ (or $f(\beta) > 0$) if $\beta > c+d-\gamma\alpha$. The last condition is equivalent to $\Re_{0\gamma} = (\beta+\gamma\alpha)/(c+d) > 1$. It is easy to check that $\det J > 0$ if and only if $\Re_{0\gamma} > 1$. This shows that the endemic equilibrium point E_+ is locally asymptotically stable if $\Re_{0\gamma} > 1$.

Now let us consider the global asymptotic stability of the endemic equilibrium point. Since the solution of (1.1) converges to $\bar{\Omega} = \{(S_1, I_1, S_2, I_2) \mid S_i + I_i = a/b\ (i = 1, 2)\}$, we now consider the following system for S_1, S_2, instead of (3.2):

$$\begin{aligned} \dot{S}_1 &= a + d\frac{a}{b} - (b+d+\alpha+\beta)S_1 + \alpha(1-\gamma)S_2 + \frac{b}{a}\beta S_1^2 + \frac{b}{a}\gamma\alpha S_2^2, \\ \dot{S}_2 &= a + d\frac{a}{b} + \alpha(1-\gamma)S_1 - (b+d+\alpha+\beta)S_2 + \frac{b}{a}\beta S_2^2 + \frac{b}{a}\gamma\alpha S_1^2. \end{aligned} \tag{3.3}$$

It is trivial that (3.3) belongs to the class called *cooperative* (note that $0 \leqslant \gamma \leqslant 1$). It is well known that planar cooperative systems do not have periodic orbits (see, for example, Smith and Waltman [13]). Further note that $dS_i/dt \leqslant 0$ for $S_i = a/b$ and $S_j \leqslant a/b$ $(i, j = 1, 2, i \neq j)$, which implies that the set $\{(S_1, S_2) \mid 0 \leqslant S_i \leqslant a/b\ (i = 1, 2)\}$ is positive invariant. Hence, we know that there is a unique endemic equilibrium point $E_+ = (S^*, S^*)$ which is locally asymptotically stable and (3.3) has no periodic orbits, if $\Re_{0\gamma} > 1$, where

$$S^* = \frac{a}{b}\frac{c+d}{\beta+\gamma\alpha} > 0.$$

Now since system (1.1) is permanent if $\Re_{0\gamma} > 1$, we can use the theory of asymptotically autonomous systems (see Castillo-Chavez and Thieme [2, Theorem 2.5]) and obtain the following theorem.

Theorem 3.5. *Suppose that $b = c$. The endemic equilibrium point E_+ is globally asymptotically stable on X_0 if $\Re_{0\gamma} > 1$.*

4. Discussion

Note that the endemic equilibrium E_+ of (1.1) exists if and only if $\Re_{0\gamma} > 1$. Theorem 2.2 implies that E_+ is locally asymptotically stable if it exists. Hence the result of Cui, Takeuchi and Saito [4] is improved by canceling their unnecessary restrictions on $\gamma\alpha$ and $\Re_0$.

Although Theorem 2.2 shows just the local asymptotic stability of E_+, the permanence result given in Theorem 3.2 implies that the disease remains endemic when $\Re_{0\gamma} > 1$. The last result does not depend on the initial conditions of (1.1), but gives the global property. Further, it is known that E_+ is locally asymptotically stable if and only if $\Re_0 > 1$, when only susceptible-individuals travel between two regions (see Cui, Takeuchi and Saito [4]). This fact can be verified if we note that $\Re_{0\gamma} = \Re_0$ when $\gamma = 0$. That is, the travel of susceptible individuals will not change the endemic condition of the regions. This strongly suggests that the restriction of the traveling of infected individuals is important to control the disease expansion.

The global asymptotic stability results of E_+ (Theorems 3.3, 3.5) are obtained for the model with some specific structures. Since we know that the endemic equilibrium point E_+ of (1.1) is, in general, locally asymptotically stable and (1.1) is permanent if $\Re_{0\gamma} > 1$, the following is our conjecture:

Conjecture. *The endemic equilibrium point E_+ of* (1.1) *is globally asymptotically stable if it exists.*

To check if the conjecture holds or not is our future problem.

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